Automatic Detection of Hopf Bifurcations on the Solution Path of a Parametrized Nonlinear Circuit

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Abstract—**A numerical technique for the automatic detection of Hopf bifurcations on the solution path of a continuously parametrized nonlinear circuit operating in time-periodic steady state is discussed. The algorithm is based on the piecewise harmonic-balance technique, is truly general-purpose, and can be applied to any kind of nonlinear microwave subsystem without restrictions on circuit topology and device models. This new software tool is of key importance in the solution of complex CAD problems such as the detection of spurious tones in any nonlinear circuit, and the analysis of the injection locking of oscillators.**

I. Introduction

LET US consider a nonlinear circuit operating in a time-periodic steady-state regime of fundamental (angular) frequency ω_S, described by some state vector $\boldsymbol{X}$. Such steady state may represent the circuit response to a periodic excitation (forced case), or be the result of a self-excited oscillation (autonomous case). If the circuit is continuously dependent on a real parameter p, the locus $\boldsymbol{X}(p)$ is a curve in the state space which will be named the *periodic solution path.* A generic natural frequency of the state $\boldsymbol{X}(p)$ will be denoted by $\sigma(p)+j\omega(p)$. If the real parts of two complex conjugate natural frequencies change sign when the parameter is swept across some critical value p_H, so that $\sigma(p_H)=0$ with $\omega(p_H)\neq 0$, then $\boldsymbol{X}(p_H)$ is a *Hopf bifurcation* of the solution path [1]. At a Hopf bifurcation, an autonomous oscillation starts up (or is extinguished), so that a further branch of the solution path originates in it. The states represented by points of the bifurcated branch are quasi-periodic [2].

From the engineering viewpoint, the Hopf bifurcation concept is of paramount importance for establishing the behavior of any kind of nonlinear subsystem. A number of complex simulation problems such as the search for spurious tones in oscillators or nonlinear amplifiers, and the determination of the stable locking range of injection-locked oscillators, can be traced back to the detection of Hopf bifurcations on the periodic solution paths of such circuits. The present Letter introduces for the first time a numerical algorithm for the automatic execution of this task.

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The fastest way of analyzing a nonlinear circuit operating in time-periodic steady state is to use the piecewise harmonic-balance (HB) technique [3], [4]. As a logical follow-up, the search for the Hopf bifurcations has also been implemented in an HB environment. This ensures general-purposeness in the broadest sense, and full compatibility with the frequency-domain characterization of the passive subnetwork by field-theoretical methods, which is necessary for the accurate simulation of microwave integrated circuits.

II. Description Of The Algorithm

In order to establish the conditions for the startup of a free oscillation, a perturbation analysis of the periodic steady state is carried out. At frequency ω, the linear subnetwork is described by the ordinary frequency-domain equations

$$[\mathbf{1}-\boldsymbol{S}(\omega)]\boldsymbol{V}(\omega)-[\mathbf{1}+\boldsymbol{S}(\omega)]\boldsymbol{I}(\omega)+\boldsymbol{F}(\omega)=\mathbf{0}, \quad (1)$$

where $\boldsymbol{V}(\omega),\boldsymbol{I}(\omega)$ are vectors of normalized voltage and current harmonics at the ports. The scattering matrix $\boldsymbol{S}(\omega)$ and the normalized forcing term $\boldsymbol{F}(\omega)$ are obtained from a linear analysis. Let us assume that for $p=p_H$ an oscillation of vanishingly small amplitude and fundamental frequency ω_H be superimposed on the steady state. The electrical regime is then quasi-periodic [4] with spectral lines at the steady-state harmonics $k\omega_S$ and at the sidebands $k\omega_S+\omega_H(-N\leq k\leq N)$. If $\Delta\boldsymbol{V},\Delta\boldsymbol{I}$ are vectors containing all the normalized sideband harmonics, the perturbation equations of the linear subnetwork can be written

$$[\mathbf{1}-\boldsymbol{S}_L]\Delta\boldsymbol{V}-[\mathbf{1}+\boldsymbol{S}_L]\Delta\boldsymbol{I}=\mathbf{0}, \quad (2)$$

where $\boldsymbol{S}_L$ is a block-diagonal matrix whose diagonal blocks are given by $S(\omega)$ computed at the sidebands. Furthermore, in the neighborhood of the steady state the nonlinear subnetwork can be described by the frequency-conversion equations [4]

$$\boldsymbol{P}^{-1}\Delta\boldsymbol{V}=\boldsymbol{Q}^{-1}\Delta\boldsymbol{I}, \quad (3)$$

where $\boldsymbol{P},\boldsymbol{Q}$ are conversion matrices which can be computed by the algorithms discussed in ref. [4]. By combining (2) and (3), we obtain the numerical definition of Hopf bifurcation:

$$D(p_H,\omega_H)\triangleq\det\{[\mathbf{1}-\boldsymbol{S}_L]\boldsymbol{P}-[\mathbf{1}+\boldsymbol{S}_L]\boldsymbol{Q}\}=0. \quad (4)$$

The best way to illustrate the strategy adopted for solving (4) is to refer to a difficult real-world engineering problem, such

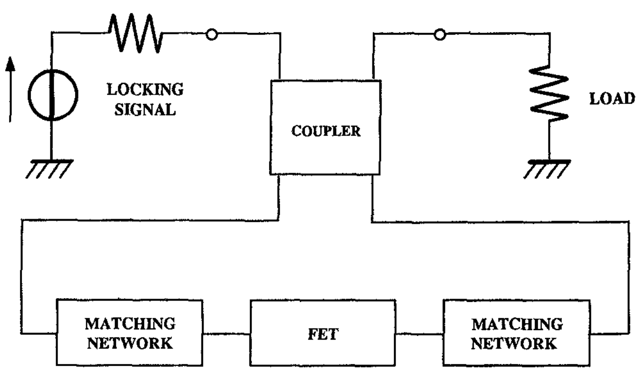

Fig. 1. Schematic topology of a two-port microwave oscillator.

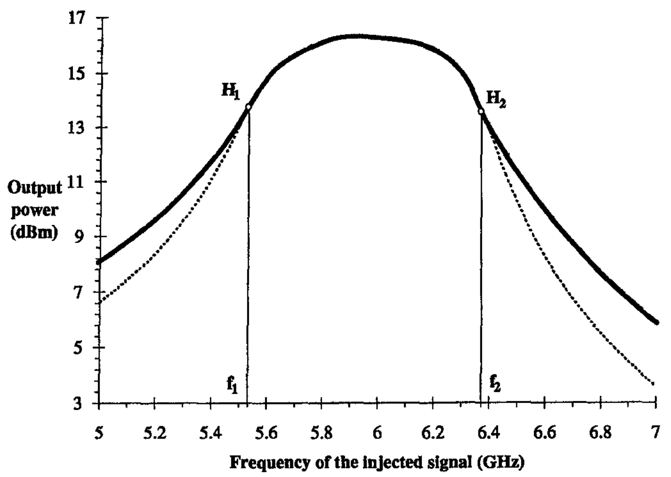

Fig. 2. Solution path for the oscillator shown in Fig. 1. Quantity plotted in the ordinate is the output power at f_i. Solid lines: periodic solutions. Dotted lines: quasi-periodic solutions [3].

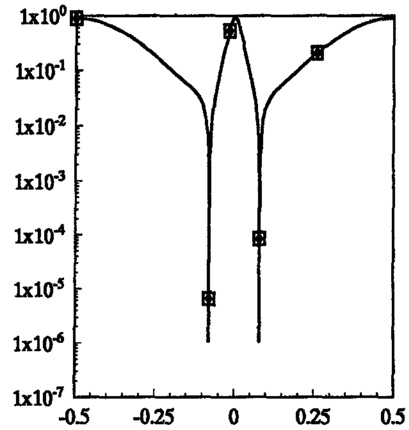

Fig. 3. Nyquist stability plots in the vicinity of H_1 (normalized magnitude). □ $f_i = 5.532319$ GHz. ♦ $f_i = 5.532317$ GHz.

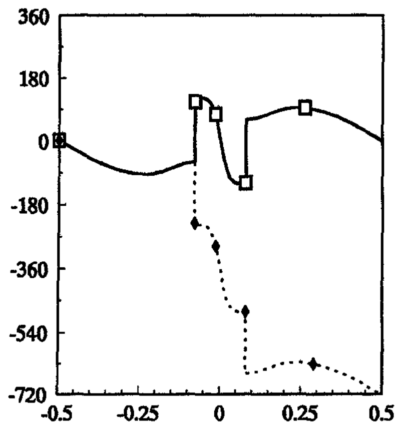

Fig. 4. Nyquist stability plots in the vicinity of H_1 (phase). □ f_i = 5.532 319 GHz. ♦ $f_i = 5.532317$ GHz.

as the determination of the stable locking range for the two-port microwave oscillator depicted in Fig. 1. It has been shown experimentally [5] that this kind of circuit may exhibit broad locking ranges, as required for instance in active phased array applications. The circuit is numerically designed [2] to support a free-running oscillation at $f_0 = 6$ GHz with an output power $P_0 \geq +15$ dBm, and it is desired to find its stable locking range when a sinusoidal signal of available power $P_i = -3$ dBm is injected at the input port.

The first step is to determine the periodic solution path by a continuation method coupled with the Newton-iteration based HB technique [3]. The obvious choice of the parameter in this case is the frequency $f_i = \omega_i/2\pi$ of the injected signal. For the oscillator of Fig. 1 the periodic solution path for $\omega_S = \omega_i$ is shown in Fig. 2 in the range 5 GHz $\leq f_i \leq$ 7 GHz. The output power at the fundamental is used in the figure as a quantity synthetically representative of the system state. As a second step, a coarse search for the Hopf bifurcations is carried out on the solution path by testing a number of uniformly spaced points $\boldsymbol{X}(p_m)$ (20 in the present application) for stability of the associated steady states. To understand this point, we observe that according to the previous discussion, the equation $D[p, \omega(p) - j\sigma(p)] = 0$ represents the characteristic equation for the natural frequencies of the state $\boldsymbol{X}(p)$ [2]. The number of unstable natural frequencies ($\sigma > 0$) can thus be computed by the Nyquist stability criterion, i.e., by counting the clockwise encirclements of the origin made by the complex quantity $D(p, \omega)$ for $-\infty < \omega < \infty$.

In practice, this task is made easy by the fact that $D(p, \omega)$ is a periodic function of ω of period ω_i [2], so that only the range $-\omega_i/2 \leq \omega \leq \omega_i/2$ has to be scanned. A Hopf bifurcation is thus detected when the quantity

$$\Delta(p_m) \triangleq \arg[D(p_m, \omega_i/2)] - \arg[D(p_m, -\omega_i/2)] \quad (5)$$

suddenly changes by 4π when stepping from p_m to $p_m + 1$, since this denotes the appearance (or disappearance) of two complex conjugate natural frequencies with positive real parts. An example is given in Fig. 4.

At this stage, a fine search for the Hopf bifurcation between p_m and p_{m+1} may be carried out by the same stability criterion and any efficient one-dimensional search algorithm. For the oscillator under consideration, two Hopf bifurcations H_1, H_2 (see Fig. 2) are found in this way on the solution path. The corresponding parameter values are $p_{H_1} = f_1 \approx 5.532318$ GHz and $p_{H_2} = f_2 \approx 6.372656$ GHz. The stable locking range is then given by $\Delta f = f_2 - f_1 \approx 840$ MHz. The Nyquist stability plots for two steady states located in the vicinity of H_1 (at $p = p_{H1} \pm$ 1 kHz) are shown in Figs. 3 and 4. These plots show the existence of two complex conjugate zeroes of D in the close neighborhood of H_1. The computation is automatic, and takes about 340 seconds overall for the case considered on an HP 9000/750 workstation. $N = 13$ harmonics are used in each HB analysis.

The well-known formula due to Adler can now be used to *define* an equivalent Q of the oscillator in the form [6]

$$Q \triangleq \frac{2f_0}{\Delta f}\sqrt{\frac{P_i}{P_0}}, \tag{6}$$

where P_i is the injected power. For the present case, (6) yields $Q \approx 1.80$. Such low values of Q are typical of this class of circuits [5].

III. Conclusion

The combination of modern harmonic-balance techniques with the principles of the mathematical theory of bifurcations can lead to the development of a new family of software tools for the rigorous CAD solution of complex engineering problems that could previously be solved only by drastic approximations. As a relevant example, the algorithm for the search of Hopf bifurcations reported in the present letter allows the automatic computation of the stable locking range of a microwave oscillator, without requiring any simplifying assumption on the oscillator model.

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